Pedal Triangles and a Fundamental Result

David Hornbeck

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Given any triangle $\triangle ABC$ and a point P, the *pedal triangle* is the triangle with vertices at the intersections of the lines through P and perpendiculars to $\overline{AB}, \overline{AC}$, & $\overline{BC}$. The pedal triangle can be inside or outside of $\triangle ABC$ depending on the position of P.

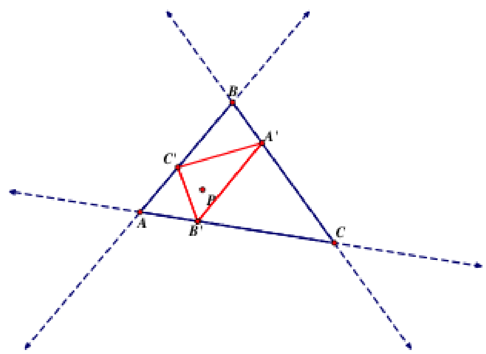
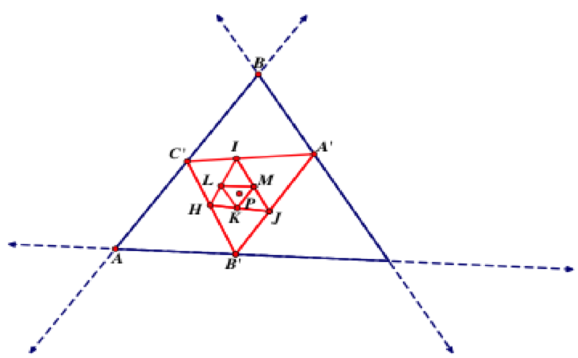


Figure 1: $\triangle A'B'C'$ is the pedal triangle of $\triangle ABC$ with pedal point P.

We are going to prove that *the pedal triangle of the pedal triangle of the pedal triangle* of a triangle is *similar* to the original triangle. We will extend and shift our diagram above:



We first claim that the pedal point P lies on the circumcircle of the following triangles:

$$\triangle AB'C', \triangle A'B'C, \triangle A'BC', \triangle B'HJ', \triangle C'HI, \triangle A'JI, \triangle ILM, \triangle HKL, \triangle JKM$$

As the proofs of each of these are similar, we will here only justify for $\triangle A'B'C$, *WLOG*.

Claim: P is on the circumcircle of $\triangle A'B'C$.
Proof: Construct the unique diameter of the circumcircle with one endpoint C. Let the other endpoint be D. Then $\angle DA'C = 90°$ and hence $\overline{DA'} \perp \overline{A'C}$. Similarly, $\angle DB'C = 90°$ and $\overline{DB'} \perp \overline{B'C}$. There is only one point D such that $\overline{DA'} \perp \overline{A'C}$ and $\overline{DB'} \perp \overline{B'C}$, though, and by definition and construction of the pedal triangle points A' & B', P is this point.

Let us now construct segments $\overline{AP}, \overline{BP}, \overline{CP}$. Then by the fact that angles that subtend the same arcs are congruent (and by the renaming of angles) [see **Figure 2**], we have

$$\angle MKP \cong \angle MJP \cong \angle IJP \cong \angle IA'P \cong \angle C'A'P \cong \angle C'B'P \cong \angle ABP$$

$$\angle LKP \cong \angle LHP \cong \angle IHP \cong \angle IC'P \cong \angle A'C'P \cong \angle A'B'P \cong \angle CBP$$

$$\Rightarrow m\angle MKL = m\angle MKP + m\angle LKP = m\angle ABP + m\angle CBP = m\angle ABC$$

Similarly, $\angle MLK \cong \angle ACB$, $\angle LMK \cong \angle BAC$.

Thus, by AAA Similarity, $\triangle MKL \cong \triangle ABC$.

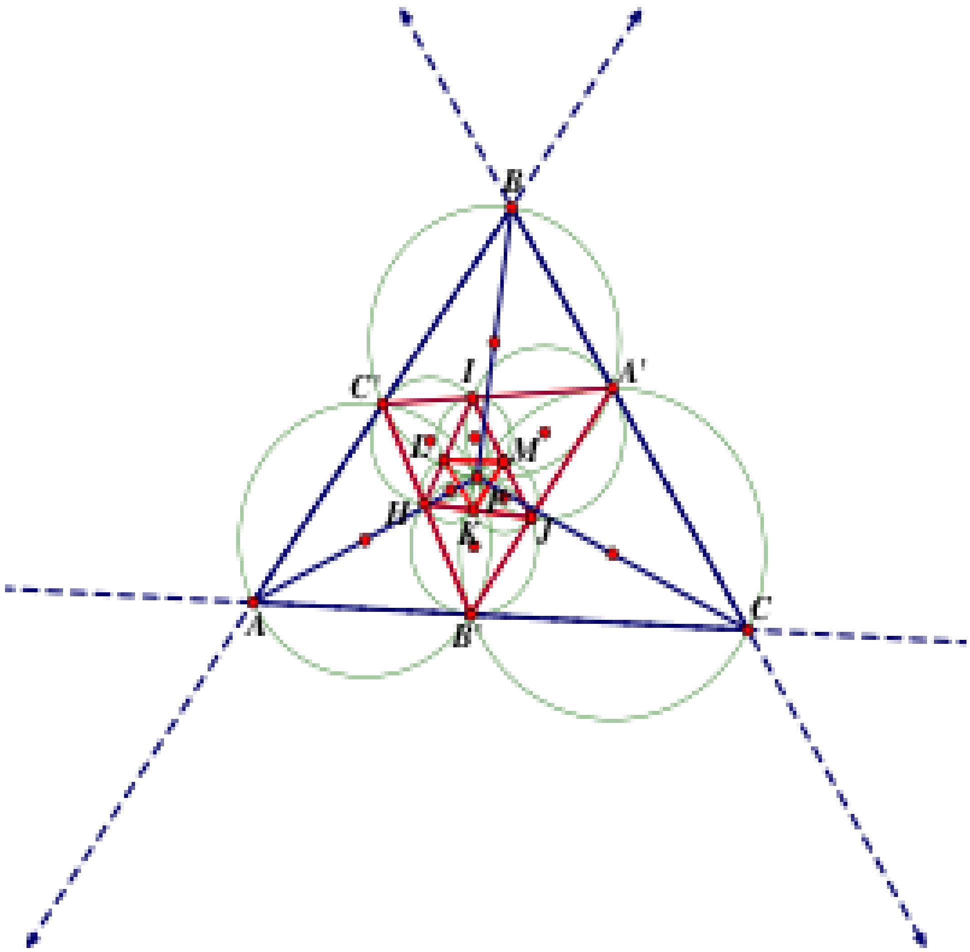


Figure 2: All of the circumcircles constructed.